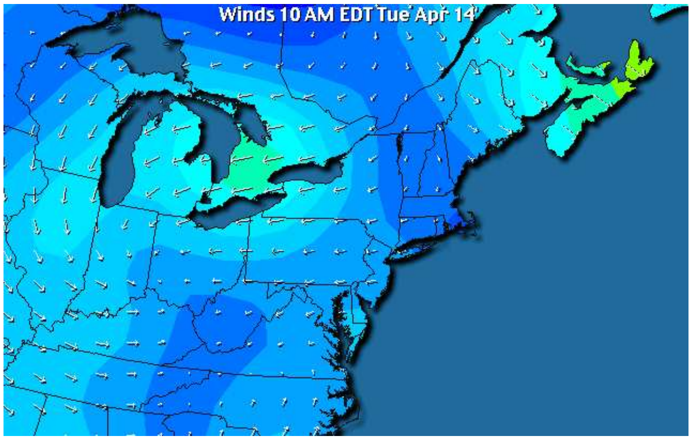
Vector Fields and Line Integrals

Here is a weather map showing the wind velocity at various points in the Northeastern United States at 10 am on April 14. This is an example of a vector field (representing velocity). If we wanted to write it using mathematical notation, we could let $\vec{F}(x, y)$ be the velocity of the wind at a point (x, y) on the map.



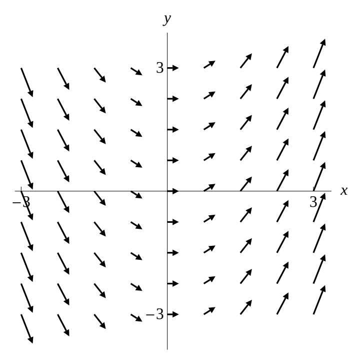
1. Match the following vector fields on $\mathbb{R}^2$ with their plots.

   (a) $\vec{F}(x, y) = \langle x, 1 \rangle$.

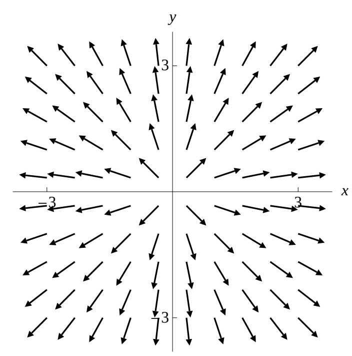
   (b) $\vec{F}(x, y) = \langle 1, x \rangle$.

   (c) $\vec{F} = \nabla f$, where f is the scalar-valued function $f(x, y) = x^2 + y^2$.

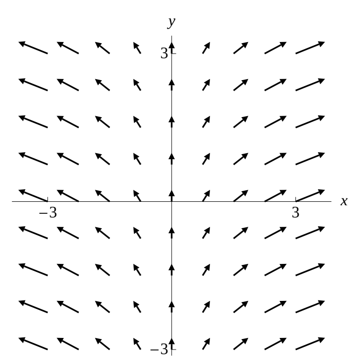
   (d) $\vec{F}(x, y) = \left\langle \dfrac{x}{\sqrt{x^2 + y^2}}, \dfrac{y}{\sqrt{x^2 + y^2}} \right\rangle$.



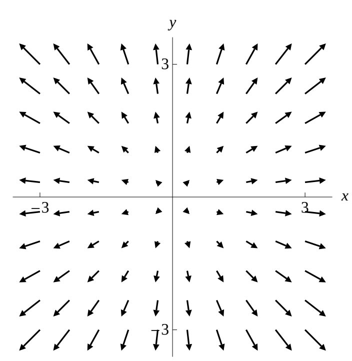
(I)



(II)

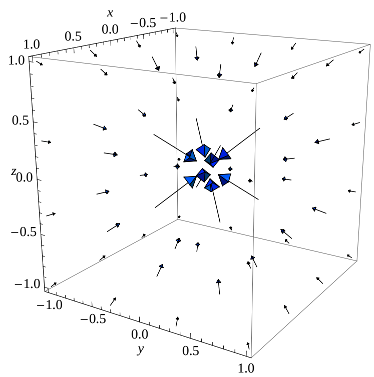


(III)

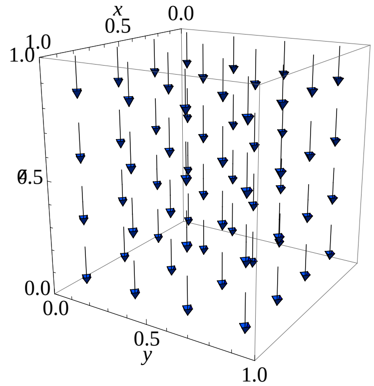


(IV)

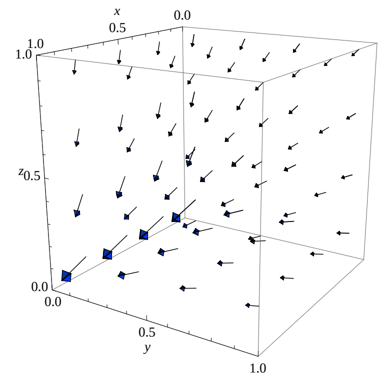
2. Match the following vector fields on $\mathbb{R}^3$ with their plots.



(I)



(II)



(III)

(a) $\vec{F}(x, y, z) = \langle 0, 0, -1 \rangle$.

(b) $\vec{F}(x, y, z) = \left\langle 0, -\frac{y}{y^2 + z^2}, -\frac{z}{y^2 + z^2} \right\rangle$.

(c) $\vec{F}(x, y, z) = \left\langle -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}, -\frac{y}{(x^2 + y^2 + z^2)^{3/2}}, -\frac{z}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle$.

3. Vector fields are used to model various things. For each of the following descriptions, decide which of the vector field plots in #2 (I, II, or III) gives the most appropriate model.

(a) Force of gravity experienced by a fly in a room. More precisely, $\vec{F}(x, y, z)$ is the force due to gravity experienced by a fly located at point (x, y, z) in a room. (Remember that force is a vector.)

(b) Force of Earth's gravity experienced by a space shuttle. More precisely, $\vec{F}(x, y, z)$ is the force that Earth's gravitational field exerts on a space shuttle located at the point (x, y, z). In the picture you've chosen, where is the Earth?

(c) ∇f, where $f(x, y, z)$ is the temperature in a room in which there is a heater along one edge of the floor. In the picture you've chosen, where is the heater? (Hint: The gradient of a function f always points in the direction in which f is ______________?)

4. Let $\vec{F}$ be the vector field on $\mathbb{R}^2$ defined by $\vec{F}(x,y) = \langle 1, x \rangle$. (We saw this vector field already in #1.)

(a) Let C be the bottom half of the unit circle $x^2 + y^2 = 1$ (in $\mathbb{R}^2$), traversed counter-clockwise. Evaluate $\int_C \vec{F} \cdot d\vec{r}$.

Note: This line integral can also be written as $\int_C 1\ dx + x\ dy$.

(b) We write $-C$ to mean the same curve as C (in this case, the bottom half of the unit circle) but oriented in the opposite direction (so clockwise instead of counter-clockwise). What is $\int_{-C} \vec{F} \cdot d\vec{r}$?

(c) Now, let C be the line segment from $(0,0)$ to $(0,1)$. Looking at the picture of $\vec{F}$ (in #1), do you think $\int_C \vec{F} \cdot d\vec{r}$ is positive, negative, or zero? Why?

(d) What if C is instead the line segment from $(0,0)$ to $(1,1)$? Is the line integral $\int_C \vec{F} \cdot d\vec{r}$ positive, negative, or zero?

5. Let $f(x,y) = e^x + xy$ and $\vec{F} = \nabla f$, a vector field on $\mathbb{R}^2$. Let C be the curve in $\mathbb{R}^2$ parameterized by $\vec{r}(t) = \langle t, t^2 \rangle$, $0 \leq t \leq 1$.

   (a) Compute the line integral $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.

   (b) What is $f(\vec{r}(t))$? Did you use this anywhere when you computed the line integral in (a)? Can you explain why this happened?

   (c) Suppose we want to look at a new curve C, parameterized by $\vec{r}(t) = \left\langle (\sin t)e^{\cos t^2 + \sqrt{t}}, \sin t + \cos t \right\rangle$ with $0 \leq t \leq \pi$. Find $\displaystyle\int_C \vec{F} \cdot d\vec{r}$.